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TIME-SINGULAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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TIME-SINGULAR SYSTEMS
OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper treats initial boundary value problems for first order linear systems of partial differential equations which are singular in time. Time singular systems are characterized by having some equations in the system impose time invariant constraints while other equations describe the time evolution of certain variables. The equations of motion for an incompressible inviscid fluid are an example. They have the constraint that the divergence of the velocities is zero, and the remaining equations govern the time evolution of the velocities. We treat both the Cauchy problem and the initial boundary value problem for these systems. We show how many boundary conditions to specify as well as a prescription to determine the well-posedness of the boundary conditions. To prove well-posedness we use normal mode analysis and pseudo-differential operators to obtain the necessary estimates. Examples of how to apply the theory are presented.

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I. Introduction

In this paper we consider linear systems of partial differential equations of the form

$$Sw_t = \sum_{j=0}^{n} A_j w_{x_j} + Bw + F(t,x) \tag{1.1}$$

where S is a singular matrix. We consider the system (1.1) for t positive and $(x_0,\cdots,x_n)$ in a sub-domain of $\mathbb{R}^{n+1}$. The dependent variable w is a vector of dimension k and each of the coefficient matrices is a complex $k \times k$ matrix. We now define precisely what is meant by a time-singular system.

Definition 1.1 The system of equations (1.1) is a time-singular system if

a) S is a singular matrix of rank $r > 0$,

b) the polynomial

$$p(s,\xi) = \det|Ss - iA\cdot\xi| \tag{1.2}$$

has degree h in s for all values of ξ in $\mathbb{R}^n - \{0\}$, and $k - h = 2e$, an even integer,

c) the roots $s(\xi)$ of $p(s,\xi) = 0$ are purely imaginary for $\xi \in \mathbb{R}^{n+1}$, and

d) there are smooth matrix functions $P(\xi)$ and $Q(\xi)$ such that

$$P(\xi)(Ss - iA\cdot\xi)Q(\xi) = \begin{pmatrix} sI_h - H(\xi) & 0 \\ 0 & Ns + E(\xi) \end{pmatrix}$$

where $H(\xi)$ is a diagonal $h \times h$ matrix and $E(\xi)$ and N are lower triangular $2e \times 2e$ matrices, N being strictly lower triangular. Also $P(\xi)$ and $Q(\xi)$ along with their inverses are bounded in norm, independent of ξ.

The lower order terms of equation (1.1) will be said to be admissible if the polynomial

$$\tilde{p}(s,\xi) = \det \left| Ss - iA\cdot\xi - B \right|$$

also has degree h in s for $|\xi| > R$ for some value of R.

One way that time-singular systems arise in applications is as the limit of hyperbolic systems that have some very large characteristic speeds. By taking the limit as the large speeds become infinite one can obtain a time-singular system. Kreiss [4] has studied hyperbolic systems that have different characteristic speeds.

Time-singular systems have both a hyperbolic character and an elliptic character. The integer h, giving the degree in s of the polymial $p(s,\xi)$ (equation (1.2)), is a measure of the hyperbolicity of the system and the integer $h-k$ is a measure of the ellipticity.

An example of a time-singular system is the following system of differential equations

$$\begin{aligned} u_t &= au_x + bu_y + p_x + cv + f_1(t,x,y) \\ v_t &= av_x + bv_y + p_y - cu + f_2(t,x,y) \\ 0 &= u_x + v_y + f_3(t,x,y). \end{aligned} \tag{1.3}$$

We have

$$\tilde{p}(x,\xi) = \det \begin{vmatrix} s-ia\xi_1-ib\xi_2 & c & i\xi_1 \\ -c & s-ic\xi_1-ib\xi_2 & i\xi_2 \\ i\xi_1 & i\xi_2 & 0 \end{vmatrix}$$

$$= (s-ia\xi_1-ib\xi_2)(\xi_1^2+\xi_2^2)$$

$$= p(s,\xi).$$

Also,

$$P(\xi) = \begin{pmatrix} \xi_2/|\xi| & -\xi_1/|\xi| & 0 \\ 0 & 0 & 1 \\ \xi_1/|\xi| & \xi_0/|\xi| & 0 \end{pmatrix},$$

$$Q(\xi) = \begin{pmatrix} \xi_2/|\xi| & \xi_1/|\xi| & 0 \\ -\xi_1/|\xi| & \xi_2/|\xi| & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For other examples of time-singular systems used in applications we refer to Oliger and Sundström [8].

The above example is a linearized, constant coefficient model of the equations for an inviscid, incompressible fluid. The equations for an inviscid, incompressible fluid can be viewed as a limiting case of the more general compressible inviscid flow equations, the limit being taken as the sound speed becomes infinite.

Of course, in applications one is frequently interested in systems with variable coefficients. However, whereas for many types of differential equations the results obtained for the case of constant coefficients extend readily to the case of variable coefficients, for time-singular systems of partial differential equations this is not the case. The effects of variable coefficients can significantly alter the behavior of the whole system. Similar behavior is described by Kreiss in [4]. As an example we present the system.

$$
\begin{aligned}
&u_x + i\,u_y + a(v_x + i\,v_y) = f_1(t,x,y) \\
&u_t + a\,v_t + v_x - i\,v_y = f_2(t,x,y).
\end{aligned}
\tag{1.4}
$$

If the coefficient a is constant, then by changing to the variable $u' = u + a\,v$, one easily sees that (1.3) is a time-singular system. However, if the coefficient a is variable, say $a = b\,x$, then the system (1.4) is seen to be equivalent to the equation

$$u'_t + (1/b)(u'_{xx} + u'_{yy}) = f_3(t,x,y) \quad ,$$

where $u' = u + a\,v$, and this is ill-posed as a Cauchy problem when b is positive.

In spite of the above example, it appears that the methods developed in this paper can be used to treat particular variable coefficient problems. But at present these methods would have to be applied on a case-by-case basis. The author conjectures that the results for the constant coefficient anelastic system can be extended to the variable coefficient systems that arise in fluid dynamics, (Oliger and Sundström [8]).

This paper is motivated by the desire to extend the results obtained by Kreiss [3] and Agmon, Douglis, and Nirenberg [1], for hyperbolic and elliptic systems of equations to other initial boundary value problems, particularly those that arise in fluid dynamics. Previously the author extended the methods employed by Kreiss to incompletely parabolic systems, (Strikwerda [10]). Incompletely parabolic systems arise in the study of viscous compressible motion. This paper extends this theory to the study of inviscid incompressible fluid motion. For a more general discussion of initial boundary value problems of fluid dynamics we refer to Oliger and Sundström [8].

As far as the author is aware of this is the only treatment of the initial boundary value problem for general time-singular systems. The Cauchy Problem for the non-linear inviscid incompressible flow equations has been studied by numerous authors, see e.g. Milne-Thomson [6].

For a treatment of a particular initial boundary value problem for an inviscid incompressible fluid see Judavič [2].

We now outline the course of this paper. In the next section we briefly develop a theory of time-singular pseudo-differential operators. We then consider the Cauchy problem for time-singular systems and then the initial boundary value problem. We show how many boundary conditions must be applied and we present a procedure to determine if a set of boundary conditions is well-posed. Finally, we consider a special important case, the linearized constant coefficient, ideal fluid flow equations and determine well-posed boundary conditions.

II. Time-Singular Pseudo-Differential Operators

In this section, we briefly develop a theory of time-singular pseudo-differential operators. The theory will be analogous to the usual theory of pseudo-differential operators and many of the results follow immediately from the usual theory. Our presentation will follow Taylor [11], also see Nirenberg [7] and Strikwerda [10].

We first define three classes of symbols of pseudo-differential operators. For convenience we define

$$<s,\xi> = (|s|^2 + |\xi|^2)^{\frac{1}{2}} \quad ,$$

$$\mathbb{C}^+(\bar{\eta}) = \{s \in \mathbb{C} : \quad Re\ s \geq \bar{\eta}\} \quad ;$$

s will always be a complex number, $s = \eta + i\tau$.

Also

$$D_x = (-i\partial_{x_0}, \cdots, -i\partial x_n)$$

where $\partial_{x_0} = \partial/\partial_{x_0}$, etc.

Definition 2.1 For m, $q \in \mathbb{R}$,

1) S^q is the set of functions $p(x,\xi)$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β there is a constant $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta} <1,\xi>^{q-|\beta|} \quad .$$

2) Sp^m is the set of functions $p(t,x,s,\xi)$ in $C^\infty(\mathbb{R}^t \times \mathbb{R}^n \times \mathbb{C}^n(\eta) \times \mathbb{R}^n)$ such that for all multi-indices α and β there is a constant $C_{a,\alpha,b,\beta}$ such that

$$|D_t^a D_x^\alpha D_\tau^b D_\xi^\beta p(t,x,s,\xi)| \leq C_{a,\alpha,b,\beta} <s,\xi>^{m-b-|\beta|} \quad .$$

3) $Ss^{m,q}$ is the set of functions $p(t,x,s,\xi)$ in $C^\infty(\mathbb{R}^t \times \mathbb{R}^n \times \mathbb{C}^n(\eta) \times \mathbb{R}^n)$ such that for all multi-indices α and β and all non-negative integers a and b there is a constant $C_{a,\alpha,b,\beta}$ such that

$$|D_t^a D_x^\alpha D_\tau^b D_\xi^\beta p(t,x,s,\xi)| \le C_{a,\alpha,b,\beta} \langle s,\xi\rangle^{m-b} \langle 1,\xi\rangle^{q-|\beta|} \quad .$$

The Fourier transform of a function $u(x)$ is

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}} e^{-i\xi\cdot x} u(x)\, dx \quad ,$$

and for a function $u(t,x)$ its Laplace-Fourier transform is

$$\hat{u}(s,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}} \int_0^\infty e^{-st} e^{-i\xi\cdot x}\, u(t,x)\, dt\, d\xi \quad .$$

We now define the corresponding pseudo-differential operators.

<u>Definition 2.2</u> If $p(t,x,s,\xi)$ is in either Sp^m or $Ss^{m,q}$ for some values of m and q then the pseudo-differential operator $p(t,x,\partial_t,D_x)$ is defined by

$$p(t,x,\partial_t,D_x)u(t,x) = \frac{1}{2\pi\,(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} p(t,x,s,\xi) e^{i\xi\cdot x} e^{st} \hat{u}(s,\xi) d\tau\, d\xi .$$

$p(t,x,\partial_t,D_x)$ is in PSp^m or $PSs^{m,q}$ if $p(t,x,s,\xi)$ is in Sp^m or $Ss^{m,q}$, respectively. The operators $PSs^{m,q}$ are called time-singular pseudo-differential operators and those in PSp^m are called pseudo-differential operators with the parameter $\eta = \mathcal{R}e\, s$.

If $p(x,\xi)$ is in S^q then the pseudo-differential operator $p(x,D_x)$ is defined by

$$p(x,D_x)u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} p(x,\xi) e^{i\xi\cdot x} \hat{u}(\xi)\, d\xi \quad ,$$

and $p(x,D_x)$ is in PS^q.

The theory of pseudo-differential operators in PS^q is developed in Taylor [11] and also Nirenberg [7]. For operators in PSp^n see Strikwerda [10].

We will allow symbols to be matrices and we will say the matrix is in, say $PS^{m,q}$, if all its elements are in $PS^{m,q}$.

We now define several norms and function spaces which will be used in the following sections.

$$|u|_q^2 = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle 1,\xi\rangle^{2q}\, d\xi,$$

$$\|u\|_{\eta,m,q}^2 = \int_{\mathbb{R}^{n+1}} \int_{-\infty}^{\infty} |\hat{u}(s,\xi)|^2 \langle s,\xi\rangle^{2m} \langle 1,\xi\rangle^{2q}\, d\tau\, d\xi,$$

where $s = \eta + i\tau$,

$$|u|_{\eta,m,q}^2 = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |\hat{u}(s,\omega)|^2 \langle s,\omega\rangle^{2m} \langle 1,\omega\rangle^{2q}\, d\tau\, d\omega$$

where $s = \eta + i\tau$.

Notice the norm $|\cdot|_{\eta,m,q}$ applies to $\mathbb{R}^n$ while $\|\cdot\|_{\eta,m,q}$ applies to $\mathbb{R}^{n+1}$.

We define the following function spaces

$$H^q = H^q(\mathbb{R}^n) = \{u(x) : |u|_q < \infty\}$$

$$L_\eta^{m,q}(\mathbb{R}_t \times \mathbb{R}^{n+1}) = \{u(t,x) : \|u\|_{\eta,m,q} < \infty\},$$

$$L_\eta^{m,q}(\mathbb{R}_t \times \mathbb{R}^n) = \{u(t,y) : |u|_{\eta,m,q} < \infty\}.$$

Note that

$$\| u \|^2_{\eta,0,q} = \int_0^\infty e^{-2\eta t} |u(t)|^2_q \, dt \quad .$$

If U is a subspace of $L^{m,q}_\eta$, we say U has finite spatial codimension if the spaces

$$u(t) = \{u(t) : u \in U\}$$

have finite codimension in $H^q(\mathbb{R}^n)$ for almost every value of t in $\mathbb{R}_+$.

For a vector w the transpose of w will be denoted w', and will say that w is in a particular function space if each of its components is in that function space,this slight ambiquity in notation should not cause any difficulty.

Note the following relations between the classes of pseudo-differential operators.

$$PS^q \subseteq PSs^{0,q}, \text{ and}$$
$$PSp^m \subseteq PSs^{m,0} \quad \text{for} \quad m \leq 0 \quad \text{and} \quad \eta > \eta_0.$$

Many of the results for time-singular pseudo-differential operators follow from the corresponding results for pseudo-differential operators. In particular, the formulae for adjoints and products are essentially the same. If $P(t,x,\partial_t,D_x)$ is in $PSs^{m,q}$ then its adjoint $P^*(t,x,\partial_t,D_x)$ has the symbol $P^*(t,x,s,\xi)$ and has the asymptotic expansion

$$P^*(t,x,s,\xi) \sim \sum_{\substack{\alpha \geq 0 \\ a \geq 0}} i^{|\alpha|+a} \frac{1}{a!\alpha!} D^\alpha_\xi D^\alpha_x D^a_\tau D^a_t P'(t,x,s,\xi) \quad .$$

where the prime on P' represents the matrix transpose.

Also if $P \in PSs^{m_1,q_1}$ and $Q \in PSs^{m_2,q_2}$ then the composition $P \cdot Q$ is in $PSs^{m_1+m_2,q_1+q_2}$ and has the symbol

$$P \cdot Q(t,x,s,\xi) \sim \sum_{\substack{\alpha \geq 0 \\ a \geq 0}} i^{|\alpha|+a} \frac{1}{a!\alpha!}(D_\tau^a D_\xi^\alpha P)(D_t^a D_x^\alpha Q) \quad .$$

The next theorem is not needed in the remainder of paper, but it should be of use in extending the results of section 4 to variable coefficient systems.

<u>Theorem 2.1</u> (Garding's Inequality for time-singular pseudo-differential operators).

If $P(t,x,\partial_t,D_x)$ is in $PSs^{-2q,2q}$ for $q \geq 0$ and $P(t,x,s,\xi) = P(t,s,s,\xi)' \geq c_0(\langle 1,\xi\rangle/\langle s,\xi\rangle)^{2q}$ then for each positive value of ε and r, there is a positive constant c_r^ε such that

$$\mathrm{Re}(w,P(t,x,\partial_t,D_x)w) \geq (c_0-\varepsilon)\|w\|^2_{\eta,-q,q} - c_r^\varepsilon \|w\|^2_{\eta,-q,q-r}$$

And moreover, for $\eta_o > 0$ there exist a subspace U of $L_{\eta_o}^{-2q,2q}$ such that U has finite spatial codimension and there is a constant c_0' such that for w in U

$$\mathrm{Re}(w,P(t,x,\partial_t,D_x)w) \geq c_0'\|w\|^2_{\eta,-q,q} \quad .$$

<u>Proof</u>

Let $\delta = \langle 1,\xi\rangle/\langle s,\xi\rangle$. Since $P = P'$ we have that for some constant M the symbol

$$P_0(t,x,s,\xi) = \tfrac{1}{2}(P^*+P) - (c_0-\varepsilon)\delta^{2q} + M\delta^{2q}\langle 1,\xi\rangle^{-1}$$

is positive definite and bounded below by $\frac{1}{2}\varepsilon\delta^{2q}$. Let $B_0(t,x,s,\xi)$ be the positive square root of P_0, then we have

$$c_1\delta^q \le B_0(t,x,s,\xi) \le C_1\delta^q \quad .$$

Now define the symbols P_i, B_i for $i > 0$ by

$$\tfrac{1}{2}(P+P^*) = (c_0-\varepsilon)\delta^{2q} + (B_0^*+,\cdots,+B_{i-1}^*)(B_0+,\cdots,+B_{i-1}) + P_i$$

and

$$B_0B_i + B_iB_0 = P_i \quad .$$

We have $P_i, B_i \in Ss^{-q,q-i}$, and $B^* - B_i \in Ss^{-q,q-i-1}$. For $i \ge r$, we have the first inequality of the theorem.

For the second part of the theorem, we rewrite the first inequality as

$$\mathrm{Re}(w,Pw) \ge (c_0-\varepsilon)\|w\|^2_{\eta,-q,q} - c_r^\varepsilon \|Kw\|^2_{\eta,-q,q} \quad ,$$

where K is the operator with symbol $\langle 1,\xi\rangle^{-r/2}$.

Since K is compact as an operator on $H^q(\mathbb{R}^n)$, we can write K as

$$K = K_\mu + F_\mu$$

where $|K_\mu|_q < \mu$ and F_μ has finite rank. (See e.g. Liusternik and Sobolev [5].)

Then taking U to be the kernel of F_μ, where μ is sufficiently small, we have the second inequality of the theorem.

We also need the following definition for our subsequent work.

Definition 2.3 If N is a nilpotent operator on the finite dimensional vector space $\mathbb{C}^n$, there exists a basis so that the matrix of N has the Jordan form,

$$\text{mat } N = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_a \end{pmatrix}$$

where J_i is a $k_i \times k_i$ lower Jordan matrix, i.e. each element of J_i is zero except for those elements immediately below the main diagonal. We can assume that the integers k_i satisfy

$$k_1 \geq k_2 \geq ,\cdots, \geq k_a$$

Let $e_{11}, e_{12}, \cdots, e_{1k_1}, e_{21}, e_{22}, \cdots, e_{2k_2}, e_{3k_3}, \cdots, e_{ak_a}$ be the corresponding basis. The <u>rearranged Jordan form</u> is that obtained by reordering the basis in reverse lexiographic order

$$e_{11}, e_{21}, \cdots, e_{a1}, e_{12}, \cdots, e_{a2}, \cdots \quad .$$

We also define the <u>degree</u> of the basis element e_{ij} as $j-1$. Note also that we do not require that the non-zero elements of J_i to be unity.

III. The Cauchy Problem for Time-Singular Systems

In this section we will show that the Cauchy Problem for time-singular systems is well-posed. Since these systems exhibit the behavior of both hyperbolic and elliptic systems, we will begin by changing to a set of dependent variables in terms of which the equations split into a hyperbolic system and a time-singular elliptic system.

Consider now the system of equations (1.1) and let $P(\xi)$ and $Q(\xi)$ be as given in Definition 1.1. Since the lower order terms in equation (1.1) are assume to be admissible, we can modify $P(\xi)$ and $Q(\xi)$ so that

$$(3.1) \qquad P(\xi)(Ss-iA\cdot\xi-B)Q(\xi) = \begin{pmatrix} I_h s - H(\xi) & 0 \\ G(\xi) & Ns + E(\xi) \end{pmatrix}$$

where $H(\xi)$ is an $h\times h$ matrix, N and $E(\xi)$ are lower triangular $2e \times h$ matrix. In addition, we assume that N is in rearranged Jordan form, (Definition 2.3). Note, also by Definition 1.1, $E(\xi)$ is non-singular for $|\xi| \geq R$, for some value of R, and $G(\xi)$ is a bounded function of ξ.

Also we have $H(\xi)$ and $E(\xi)$ in S^1 and $G(\xi)$ in S^0.

Now we operate on the system (1.1) on the left with the operator $P(D_x)$, and define $\tilde{w}$ by

$$w = Q(D_x)\tilde{w} \quad .$$

If we let w^1 be the first h components of $\tilde{w}$ and w^2 be the last 2e components, then we obtain the new system

$$\partial_t w^1 - H(D_x)w^1 = \tilde{F}^1(t,x)$$

$$N\partial_t w^2 + E(D_x)w^2 = -G(D_x)w^1 + \tilde{F}^2 \quad .$$

In the case of the system (1.2), note that w^1 is essentially the vorticity $u_y - v_x$ and w^2 is essentially the divergence and the pressure p.

The first equation in the system (3.2) is a system of hyperbolic pseudo-differential equations and therefore we have the estimate (see Taylor [10]).

$$|w^1(t)|_q^2 \leq C'_q(|w^1(0)|_q^2 + \int_0^t |\tilde{F}^1(\tau)|_q^2 \, d\tau) .$$

In the η-norm this becomes

$$\| w^1 \|^2_{\eta,0,q} \leq \frac{1}{\eta} C'_q (|w^1(0)|^2_q + \| \tilde{F}^1 \|^2_{\eta,0,q}) \quad .$$

We now consider the second equation in the system (3.2). Let

$$w^2 = (w^{20}, w^{21}, \cdots w^{2d})'$$

where w^{2j} consists of those components of w^2 which have degree j as determined by the rearranged Jordan form of N, (Definition 2.3). Define the matrix $\Gamma(s,\xi)$ by

$$\Gamma w^2 = (w^{20}, \delta w^{21}, \delta^2 w^{22}, \cdots \delta^d w^{2d})'$$

where $\delta = \langle 1,\xi\rangle / \langle s,\xi\rangle$. The operator $\bar{E}$ whose symbol is given by

$$\bar{E}(s,\xi) = \Gamma(Ns - E(\xi)\Gamma^{-1}$$

has all of its elements in $PSs^{0,1}$. Moreover we can construct an operator $\bar{E}^{(-)}(\partial_t, D_x)$ in $PSs^{0,-1}$, so that

$$\bar{E}^{(-)}(s,\xi) = E(s,\xi)^{-1} \quad \text{for} \quad |\xi| \geq R.$$

and

$$\bar{E}^{(-)}(s,\xi)\bar{E}(s,\xi) = I_{2e} + K_2(s,\xi)$$

where $K_2 \in PSs^{0,-1}$. We then have

$$(I+K_2)\Gamma w^2 = \bar{E}^{(-)} \bar{E}\Gamma w^2 = \bar{E}^{(-)}\Gamma(-Gw^1 + \tilde{F}^2(t,x)) \quad .$$

We thus obtain the estimate

$$\| \Gamma w^2 \|^2_{\eta,0,q+1} \leq C(\| w^1 \|^2_{\eta,0,q} + \| \Gamma\tilde{F}^2 \|^2_{\eta,0,q}) + \| K\Gamma w^2 \|^2_{\eta,0,q+1}$$

where K is a compact operator on H^{q+1}.

Since K is compact, we can write K as $K_\varepsilon + F_\varepsilon$ where the norm of K_ε is less than ε and F_2 has finite rank, (Linsternik and Sobolev [4]). Therefore, by restricting Γw^2 to an appropriate space of finite spatial codimension, and combining the above estimate with the estimate for w^1 we obtain

$$\eta\| w^1\|_{\eta,0,q} + \| \Gamma w^2\|_{\eta,0,q+1} \tag{3.3}$$

$$\leq C(|w^1(0)|^2_q + \|\tilde{F}^1\|^2_{\eta,0,q} + \|\Gamma\tilde{F}^2\|^2_{\eta,0,q}) \quad .$$

We summarize the above computations in the following theorem.

<u>Theorem 3.1</u> The Cauchy Problem for time-singular systems is well-posed in the sense that there exist subspaces L_1 and L_2 of $L^{0,q}_\eta$, for $q \geq 0$, that have finite spatial codimension and for

$$w^1(0) \in H^q, \qquad \tilde{F}^1 \in L^{0,q}_\eta \qquad \text{and} \qquad \Gamma\tilde{F}^2 \in L_1 \quad ,$$

there is a unique solution w with

$$w^1 \in L^{0,q}_\eta \qquad \text{and} \qquad \Gamma w^2 \in L_2 \quad ,$$

and the estimate (3.3) holds.

IV. <u>The Initial Boundary Value Problem</u>

Consider the system (1.1) in the case where the coefficients are constant and where Ω is the space $\mathbb{R}_+ \times \mathbb{R}^n$. Rewriting (1.1) in the coordinates (t,x,y) where $t \geq 0$, $x \geq 0$, $y \in \mathbb{R}^n$, we obtain

$$Sw_t = A_0 w_x + \sum_{j=1}^{n} A_j w_{y_j} + Bw + F(t,x,y) \quad . \tag{4.1}$$

On the boundary $x = 0$, there are boundary conditions

$$(4.2) \qquad Tw = g(t,y)$$

and, for simplicity, the initial condition will be

$$(4.3) \qquad w(0,x,y) = 0 .$$

We assume that the boundary is non-characteristic, that is, A_0 is non-singular. We now re-write equation (4.1) as

$$(4.4) \qquad \begin{aligned} w_x &= A_0^{-1}(Sw_t - \sum_{j=1}^{n} A_j w_{y_j} - Bw) + A_0^{-1}F(t,x,y) \\ &= M(\partial_t, D_y)w + \tilde{F}(t,x,y) . \end{aligned}$$

We now examine the operator $M(\partial_t, D_y)$. Let $M_1(\partial_t, D_y)$ be the first order operator part of M, i.e.

$$M_1(\partial_t, D_y) = A_0^{-1}(S\partial_t - i \sum A_j \partial_{y_j}).$$

Definition 4.1 Let h_- and h_+ be the number of negative and positive roots, respectively, of the following equation in κ,

$$(4.5) \qquad \det |S - A_0\kappa| = 0 .$$

Notice that by Definition 1.1 the roots of equation (4.5) are real and that zero is a root of multiplicity $2e$. We now consider again the polynomial $p(s,\xi)$ as defined in Definition 1.1. We write

$$(4.6) \qquad \begin{aligned} 0 = p(s,\lambda,\omega) &= \det |S_s - A_0\lambda - i\sum_j A_j\omega_j| \\ &= s^h p_{2e}(\lambda,\omega) + s^{h-1}p_{2e+1}(\lambda,\omega) + \cdots + p_{2e+h}(\lambda,\omega) \end{aligned}$$

where $p_m(\lambda,\omega)$ is a homogeneous polynomial in (λ,ω) of degree m. We now make another assumption that is often made in the study of elliptic boundary value problems, see Agmon, Douglis, and Nirenberg [1].

Assumption 4.1 For each $\omega_0 \neq 0$, the polynomial $P_{2e}(\lambda,\omega_0)$, as a polynomial in λ, has exactly e roots with real part of λ positive.

Note that for $n > 1$, this assumption is automatically satisfied, see Agmon, et al [1]. We also make an assumption that simplifies the hyperbolic boundary value problem, see Kreiss [3].

Assumption 4.2 The roots $s(\xi)$ of $p(s,\xi) = 0$ are distinct for $|\xi| = 1$.

This last assumption can be weakened but it makes the following result easier to prove.

Theorem 4.1 The eigenvalues of the symbol $M_1(s,\omega)$ depend on (s,ω) in the following manner:

1) For $\mathcal{Re}\, s > 0$ and $\omega \in R^n - \{0\}$, there are $h_- + e$ eigenvalues with negative real part and $h_+ + e$ eigenvalues with positive real part.

2) For $\mathcal{Re}\, s \geq 0$, $|s| \geq a > 0$ and $\varepsilon \geq |\omega| \geq 0$ there are h eigenvalues bounded away from zero in absolute value and $2e$ eigenvalues λ satisfying

$$C|\omega| \geq |\lambda| > c|\omega| \quad \text{and}$$

$$|\mathcal{Re}\lambda| \geq \delta_1|\omega|$$

for some positve constants δ_1, C, and c.

3) For $|\omega| \geq a > 0$ and $\varepsilon > \mathcal{Re}\, s \geq 0$ there are at least $2e$ eigenvalues whose real parts are bounded away from zero, and the remaining eigenvalues λ satisfy

$$|\mathrm{Re}\,\lambda| \geq \delta_2\, \mathrm{Re}\, s$$

for some positive constant δ_2.

Proof If λ is an eigenvalue of $M_1(s,\omega)$ then

$$0 = \det|\lambda - M(s,\omega)| = \det A_0^{-1} \cdot \det|Ss - A_0\lambda - iA\cdot\omega| \ .$$

If λ were purely imaginary and non-zero, then by Definition 1.1, s must also be purely imaginary. So, if s has positive real part then either λ is zero or has non-zero real part. If, in addition to s having positive real part, ω is non-zero then Definition 1.1 shows that λ can not be zero. To determine the precise number of eigenvalues with positive real part we will examine $\lambda(s,\omega)$ for ω near zero. For $\omega = 0$, we have

$$0 = \det|Ss - A_0\lambda| = \det|S - A_0\frac{\lambda}{s}|\, s^k \ .$$

So $\lambda = \kappa s$ where κ is satisfies equation (4.5), thus there are h_- eigenvalues with $\mathrm{Re}\,\lambda < 0$ and h_+ eigenvalues with $\mathrm{Re}\,\lambda > 0$. To examine the remaining $2e$ eigenvalues that vanish at $\omega = 0$, we set $\omega = \varepsilon\omega'$ where $|\omega'| = 1$, and consider equation (4.6) as ε tends to zero. Let $\lambda = \varepsilon\lambda$,

$$0 = p(s,\lambda,\omega') = \varepsilon^{2e} s^h p_{2e}(\lambda',\omega') + \varepsilon^{2e+1} s^{h-1} p_{2e+1}(\lambda',\omega')$$

$$+ \cdots + \varepsilon^{2e+h} p_{2e+h}(\lambda',\omega') \ .$$

So, for ε near zero, there are $2e$ eigenvalues given by

$$\lambda = \varepsilon\, \lambda'(\omega)(1+0(\varepsilon))$$

where $\lambda'(\omega)$ satisfies $p_{2e}(\lambda'(\omega),\omega) = 0$. This, by Assumption 4.1, proves the first two parts of the theorem.

The third part of the theorem follows similarly from Assumption 4.2.

The eigenvalues of $M_1(s,\omega)$ with negative real parts when $Re\, s > 0$, correspond to that part of the solution of equation (4.1) propagating into the region $x > 0$, and therefore the correct number of boundary conditions should be $h_- + e$.

Assumption 4.3 There are exactly $h_- + e$ linearly, independently boundary conditions, that is the matrix $T(t,y)$ has rank $h_- + e$ for all values of (t,y).

We must now define two types of eigensolutions. The existence of eigensolutions for an initial boundary value problem indicates that the problem is ill-posed. The eigensolutions represent a family of solutions to the initial boundary value problem for which the norm of the solution is not bounded by the data.

Definition 4.2 An eigensolution of hyperbolic type for the initial boundary value problem (4.1) - (4.3) is a solution $w(x,s,\omega)$ to the ordinary differential equation

$$\frac{d}{dx} w = M_1(s,\omega)w$$

satisfying

a) $Re\, s \geq 0 \quad , \quad \omega \neq 0$

b) $Tw(0,s,\omega) = 0$

c) For $Re\, s > 0$, $w(x,s,\omega)$ is bounded for $x > 0$, and for $Re\, s = 0$, then

$$w(x,s,\omega) = \lim_{\varepsilon \to 0+} w(x,s+\varepsilon,\omega)$$

where for each ε, $w(x,s+\varepsilon,\omega)$ is a bounded solution on $x > 0$ of

$$\frac{d}{dx} w = M_1(s+\varepsilon,\omega).$$

We need also to define elliptic eigensolutions. But before doing so we must define two auxiliary matrix functions $\approx{M}(\omega)$ and $\approx{T}(\omega)$.

Definition 4.3 The matrices $\tilde{\tilde{M}}(\omega)$ and $\tilde{\tilde{T}}(\omega)$ are defined by the following procedure.

1. Let $P(s,\omega)$ be a smooth nonsingular matrix function in a neighborhood of $\omega = 0$, $|s| = 1$, $Re\, s \geq 0$ so that

$$M(s,\omega) = P M_1 P^{-1} = \begin{pmatrix} \tilde{M}_{11} & 0 \\ 0 & \tilde{M}_{22} \end{pmatrix}$$

is in lower triangular form and $\tilde{M}_{22}(1,0)$ is a nilpotent $2e \times 2e$ matrix in rearranged Jordan form and $\tilde{M}_{11}$ is an $h \times h$ matrix.

2. Construct the $2e \times 2e$ matrix $\tilde{\tilde{M}}(\omega)$

$$\tilde{\tilde{M}}_{ij}(\omega) = \begin{cases} M_{22,ij}(1,\omega) & \text{if} \quad d_i = d_j \\ & \text{or} \quad M_{22,ij}(1,0) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

($M_{22,ij}$ is the element of M_{22} in the i^{th} row and j^{th} column, and d_i is the degree of the i^{th} basis element of $\tilde{M}_{22}$).

3. Let

$$\tilde{T}(s,\omega) = Q_0(s,\omega) T P^{-1}(s,\omega) = \sum_{\ell=0}^{\infty} s^{-\ell} \tilde{T}^{(\ell)}(\omega)$$

where $Q_0(s,\omega)$ is a non-singular, smooth matrix function defined in a neighborhood of $\omega = 0$, $|s| = 1$, $Re\, s \geq 0$. $Q_0(s,\omega)$ is somewhat arbitrary but restricted by the condition in part 4.

4. Define the integers

$$c_i = \max\{d_j - \ell \;\; ; \;\; T_{ij}^{(\ell)}(\omega) \neq 0\}$$

then the boundary operator $\tilde{\tilde{T}}(\omega)$ is given by

$$\tilde{\tilde{T}}^{(\ell)}_{ij}(\omega) = \begin{cases} \tilde{\tilde{T}}^{(\ell)}_{ij}(\omega) & \text{if} \quad d_j - \ell = c_i \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{\tilde{T}}(\omega) = \sum_{\ell} \tilde{\tilde{T}}^{(\ell)} .$$

$Q_0(s,\omega)$ is chosen so that the rank of $\tilde{\tilde{T}}(\omega)$ is equal to the rank of T, and $Q_0(s,\omega) = Q_{00} + \mathcal{O}(\omega/s)$ where Q_{00} is non-singular.

<u>Definition 4.4</u> An eigensolution of elliptic type for the initial boundary value problem 4.1 - 4.3 consists of a vector v_0 and a function $u(x,\omega)$ such that

1) v_0 is in the span of those eigenvectors of $\tilde{M}_{11}(1,0)$ whose eigenvalues have negative real part.

2) $u(x,\omega)$ is a solution to the ordinary differential equation

$$\frac{d}{dx} u = \tilde{\tilde{M}}(\omega)u$$

where $|\omega| = 1$, and $u(x,\omega)$ is bounded for $x \geq 0$.

3) The vector $w(\omega) = (v_0, u(0,\omega))'$ satisfies

$$\tilde{\tilde{T}}(\omega)w(\omega) = 0.$$

As in the Cauchy Problem the estimates we obtain will not be in terms of the dependent variable w, but rather in variables which exhibit the mixed hyperbolic and elliptic character of the equation (4.1). Let $U(s,\omega)$ be a smooth matrix in $\mathbb{C}^+(0) \times R^n$ such that each element of $U(s,\omega)$ is in $Ss^{m,q}$ for some m and q, and

$$U(s,\omega)M_1(s,\omega)U(s,\omega)^{-1} = \begin{pmatrix} \tilde{M}_1(s,\omega) & 0 \\ \tilde{M}_3(s,\omega) & \tilde{M}_2(s,\omega) \end{pmatrix} \tag{4.7}$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are lower triangular and of dimension $h \times h$ and $2e \times 2e$, respectively, and $\tilde{M}_3(s,\omega)$ is zero for $|\omega| \leq \varepsilon \langle s,\omega\rangle$. Moreover, we require that $\tilde{M}_2(s,0)$ be nilpotent and in rearranged Jordan form, (Definition 2.3), and for $|\omega| \geq (1-\varepsilon)\langle s,\omega\rangle$ the eigenvalues of $\tilde{M}_2(s,\omega)$ are bounded away from the imaginary axis.

Such a matrix U exists because the reduction of a matrix to lower triangular form can be made a smooth function of the matrix elements, and the various conditions on $\tilde{M}_2$ and $\tilde{M}_3$ are easily seen to be possible in the light of Theorem 4.1.

Let v be the first h components of Uw and let u be the last $2e$ components. In a way similar to the Cauchy Problem we define the matrix Γ by

$$\Gamma u = (u^0, \delta u^1, \delta^2 u^2, \cdots, \delta^d u^d)'$$

where $\delta = \langle 1,\omega\rangle / \langle s,\omega\rangle$ and u^j is the vector containing the components of u which are of degree j as determined by the rearranged Jordan form of $\tilde{M}_2(s,0)$.

With these new dependent variables we rewrite (4.4) as

$$\begin{aligned} v_x &= \tilde{M}_1\ (\partial_t, D_y)v + \tilde{F}_1(t,x,y) \\ u_x &= \tilde{M}_3\ (\partial_t, D_y)v + \tilde{M}_2\ (\partial_t, D_y)u + \tilde{F}_2(t,x,y) \end{aligned} \tag{4.8}$$

and the boundary condition becomes

$$QT(v,u)' = Qg(t,y)$$

where $Q(\partial_t, D_y)v$ is a bounded operator given by

$$Q(s,\omega) = \tilde{\Gamma} Q_0(s,\omega)$$

where $Q_0(s,\omega)$ is as in Definition 4.3, and $\tilde{\Gamma}$ is a diagonal matrix with

$$\tilde{\Gamma}_{ii} = \delta^{c_i}$$

where c_i is as in Definition 4.3.

We now give the main theorem of this section.

Theorem 4.2 The initial boundary value problem (4.1) - (4.3) is well-posed if and only if there are no eigensolutions of either hyperbolic or elliptic type. By well-posed we mean that the following estimate holds

$$\eta \| v \|^2_{\eta,0,0} + \| \Gamma u \|^2_{\eta,0,1} + |v|^2_{\eta,0,0} + |\Gamma u|^2_{\eta,0,0}$$

(4.9)

$$\leq C(|Qg|^2_{\eta,0,0} + \| \tilde{F}^1 \|^2_{\eta,0,0} + \| \Gamma \tilde{F}^2 \|^2_{\eta,0,0} + \| \Gamma u \|^2_{\eta,0,0}) .$$

Note that the norms $|\cdot|_{\eta,0,0}$ apply to the boundary $x = 0$, and the norms $\| \cdot \|_{\eta,0,0}$ apply to the interior, $x > 0$.

Before proving Theorem 4.2 we state the following theorem which will be used to prove Theorem 4.2.

Theorem 4.3 There exists a hermitian matrix $R(s,\omega)$ defined on $\mathbb{C}^+(0) \times \mathbb{R}^n$ such that

1) $\mathrm{Re}\,(w,R(s,\omega)M(s,\omega)w) \geq \eta|v|^2 + (|\omega|-c)|\Gamma u|^2$

2) $|(w,Rf)| \leq |v|^2 + |\Gamma u|^2 + C(|f^1|^2 + |\Gamma f^2|^2)$

3) if there are no eigensolutions of either hyperbolic or elliptic type then there exists a bounded matrix $Q(s,\omega)$ such that

$$(w,(R+(QT)^t(QT))w) \geq c_0(|v|^2+|\Gamma u|^2$$

for some positive constant c_0.

Proof of Theorem 4.2

Define the norm $\|\cdot\|_+$ by

$$\|f\|_+^2 = \int_0^\infty |f(x)|^2 dx \quad .$$

Then by Theorem 4.3, we have for w, a solution of (4.1-4.3),

$$\eta\|v\|_+^2 + |\omega| \|\Gamma u\|_+^2 \leq \int_0^\infty \mathcal{Re}(w,R(s,\omega)M(s,\omega)w)dx$$

$$= \mathcal{Re}\int_0^\infty (w,R(s,\omega)w_x)dx - \int_0^\infty \mathcal{Re}(w,RF)dx$$

$$= -\tfrac{1}{2}(w,Rw)_{x=0} - \int_0^\infty \mathcal{Re}(w,RF)dx$$

$$\leq \tfrac{1}{2}|QTw(0)|^2 - \tfrac{1}{2}c_0(|v(0)|^2 + |\Gamma u(0)|^2)$$

$$+ \|v\|_+^2 + \|\Gamma u\|_+^2 + C(\|F^1\|_+^2 + \|\Gamma F^2\|_+^2 \quad .$$

Now integrating with respect to $\tau = \mathrm{Im}\, s$ and ω, we obtain the estimate (4.9). Therefore, if there are no eigensolutions then the problem is well-posed.

Suppose now that there were an eigensolution of hyperbolic type. Since $|\omega| \neq 0$, $\Gamma \geq \varepsilon I_{2e}$ and $|Q(s,\omega)| \geq c_1(\varepsilon)$ so we can ignore the operators

Γ and Q. If $w(x,s,\omega)$ is an eigensolution of hyperbolic type then by using cut-off functions, one can easily show that there are solutions to (4.1 - 4.3) with F and g arbitrarily small in norm and yet have $|w|_\eta \geq 1$. This shows (4.1 -4.3) to be ill-posed

Similarly, if there is an eigensolution of elliptic type, one can construct solutions to (4.1 - 4.3) such that $|w|_\eta \geq 1$, but $|F_1|$, $|\Gamma F_2|$ and $|Qg|$ all are arbitrarily small. To do this set $s = 1$ and $\omega = \varepsilon\omega_0$. Then one can check that the condition that QTw can be made arbitrarily small as ε approaches zero, with $|w|_\eta \geq 1$, is that there be an eigensolution of elliptic type.

Proof of Theorem 4.3

The construction of the matrix $R(s,\omega)$ for $\mathcal{Re}\, s \geq 0$ and $|\omega| \geq \varepsilon <s,\omega>$ is essentially the same as the construction in Kreiss [3], (see also Ralston [9]). We only point out that for $|\omega| \geq \varepsilon <s,\omega>$ we can take $Q = I$ and ignore Γ since $\varepsilon^d I \leq \Gamma \leq I$.

We now consider the case $|\omega| \leq \varepsilon <s,\omega>$. We begin with the matrix $\tilde{M}_2(s,\omega)$ as given in equation (4.7). Since $\tilde{M}_2(s,0)$ is in rearranged Jordan form the matrix

$$\Gamma \tilde{M}_2(s,\omega)\Gamma^{-1} = \tilde{\tilde{M}}_2(s,\omega)$$

has all of its off diagonal elements bounded by $<1,\omega>$. The eigenvalues of $\tilde{\tilde{M}}_2(s,\omega)$ are bounded by $|\omega|$ according to Theorem 4.1. Let $U_1(s,\omega)$ be a nonsingular smooth function of (s,ω) for $|\omega| \leq \varepsilon <s,\omega>$, such that

$$U_1 M_2 U_1^{-1} = \begin{pmatrix} N_+ & 0 \\ 0 & N_- \end{pmatrix}$$

where the eigenvalues of N_+ have positive real part and those of N_-

have negative real part, in addition N_+ and N_- are lower triangular. Then there are matrices D_+ and D_- such that

$$D_+ > 0 \quad , \quad D_- < 0 \quad , \quad \text{and}$$

$$\mathcal{Re}(D_\pm N_\pm) \ \geq \ c_1 I_e(|\omega|-c_0)$$

for some constants c_1 and c_0, $c_1 > 0$.

Then $R_2(s,\omega)$ is defined as

$$R_2 = U_1^* \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} U_1 \quad .$$

We then have $\mathcal{Re} R_2 \tilde{\tilde{M}}_2 \ \geq \ c_1'(|\omega|-c_0)I_{2e}$. Similarly, we can construct $R_1(s,\omega)$ so that

$$\mathcal{Re}\ R_1 \tilde{\tilde{M}}_1 \ \geq \ \eta\, I_h \quad ,$$

and then $R(s,\omega)$ is constructed as

$$U^*(s,\omega) \begin{pmatrix} R_1 & 0 \\ 0 & \Gamma R_2 \Gamma \end{pmatrix} U(s,\omega).$$

This proves the first inequality in Theorem 4.3. We also see that the second inequality is satisfied. To establish the third part of the theorem, note that from the above construction of R we have that

$$(w,\ Rw) \ \geq \ c_+(|v^+|^2 + |(\Gamma u)^+|^2) \ - c_-\ (|v^-|^2 + |(\Gamma u)^-|^2)$$

where v^+, (resp. v^-) is the projection of v on the subspace generated by the eigenvectors of $\tilde{M}_1$ having positive (resp. negative) real parts. Similarly for $(\Gamma u)^+$ and $(\Gamma u)^-$.

Now, the condition that there are no eigensolutions is precisely the condition that

$$|QTw^-|^2 \geq C(|v^-|^2 + |(\Gamma u)^-|^2)$$

for all (s,ω), $\mathcal{Re}\, s \geq \eta$, for some choice of the positive constant C. Therefore, we have

$$
\begin{aligned}
(w,Rw) &\geq c_+(|v^+|^2+|(\Gamma u)^+|^2) - c_-(|v^-|^2+|(\Gamma u^-|^2) \\
&\geq c_+ |v^+|^2 + |(\Gamma u)^+|^2 - c|QTw^-|^2 + |v^-|^2 + |(\Gamma u)^-|^2 \\
&\geq c_+ |v^+|^2 +|(\Gamma u)^+|^2 - c|QTw|^2 - |QTw^+|^2 \\
&\geq c'(|v^+|^2+|(\Gamma u)^+|^2 + |v^-|^2 + |(\Gamma u)^-|^2 - c|QTw|^2 \\
&> c'(|v|^2+|\Gamma u|^2) - c|QTw|^2 \qquad ,
\end{aligned}
$$

which shows that the third inequality holds. This proves Theorem 4.3.

V. The Initial Boundary Value Problem for the Linearized Ideal Fluid Equations

In this section we apply the results of the preceding section to a particular set of equations that is similar to those that frequently arise in applications. These equations are a constant coefficient version of the ideal fluid equations.

$$
\begin{aligned}
(5.1) \qquad u_t &= -a\,u_x - b\,u_y - p_x + f_1(t,x,y) \\
v_t &= -a\,v_x - b\,v_y - p_y + f_2(t,x,y) \\
0 &= u_x + v_y + f_3(t,x,y)
\end{aligned}
$$

on the region $x \geq 0$, $y \in \mathbb{R}$, $t \geq 0$. The equations are essentially the same as equation (1.3), therefore

$$p(s,\xi) = (\xi_1^2+\xi_2^2)(s+ia\,\xi_1+ib\xi_2) .$$

We see that $h = 1$ and $e = 1$, and the coefficients a and b must be real. To determine how many boundary conditions are needed at the boundary $x = 0$, we evaluate

$$\det|S - A_0\kappa| = \det \begin{vmatrix} 1+a\kappa & 0 & \kappa \\ 0 & 1+a\kappa & 0 \\ \kappa & 0 & 0 \end{vmatrix} = -\kappa^2(1+a\kappa) .$$

Thus we see, by Definition 4.1 and Assumption 4.3, if $a > 0$ then $h_- = 1$, so we need two boundary conditions, and if $a < 0$, then $h_- = 0$ and we need only one boundary condition.

To check the boundary conditions we need to look for eigensolutions of hyperbolic and elliptic type. A hyperbolic eigensolution will satisfy the ordinary differential equation

$$\begin{pmatrix} v \\ u \\ p \end{pmatrix}_x = \begin{pmatrix} -s'/a & 0 & -i\omega/a \\ -i\omega & 0 & 0 \\ a\,i\,\omega & -s' & 0 \end{pmatrix} \begin{pmatrix} v \\ u \\ p \end{pmatrix} \tag{5.2}$$

where $s' = s + i\,b\,\omega$. Two solutions, which are linearly independent for $s' \neq |\omega|a$, are

$$\begin{pmatrix} -i\omega \\ |\omega| \\ s'-a|\omega| \end{pmatrix} e^{-|\omega|x} \quad \text{and} \quad \begin{pmatrix} s' \\ i\omega a \\ 0 \end{pmatrix} e^{-s'x/a} . \tag{5.3}$$

For elliptic eigensolutions, we set

$$\tilde{v} = v + (p+au)i\omega/s' \quad ,$$

and

$$\tilde{u} = u - p\frac{|\omega|}{s'}(1+O(\omega/s')) \quad .$$

Then equation (5.2) becomes

$$\begin{pmatrix} \tilde{v} \\ \tilde{u} \\ p \end{pmatrix}_x = \begin{pmatrix} -s'/a & 0 & 0 \\ -i & \lambda_1 & 0 \\ a\,i & -s'(1+\mu) & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{v} \\ \tilde{u} \\ p \end{pmatrix}$$

where $\lambda_1 = |\omega|(1+O(\omega/s'))$, $\lambda_2 = -|\omega|(1+O(\omega/s')$ and $\mu = O(\omega/s')^2$. Then an elliptic eigensolution satisfies the ordinary differential equation

$$\begin{pmatrix} \tilde{u} \\ p \end{pmatrix}_x = \begin{pmatrix} |\omega| & 0 \\ -s' & -|\omega| \end{pmatrix}\begin{pmatrix} \tilde{u} \\ p \end{pmatrix} \quad . \tag{5.4}$$

We now consider the boundary conditions themselves.

Case 1. $a < 0$. This corresponds to an outflow boundary. We need one boundary condition, let it be

$$t_1 u + t_2 v + t_3 p = g(t,y) \quad .$$

First we look for hyperbolic eigensolutions. The only possible eigensolution for $a < 0$, is

$$\begin{pmatrix} v \\ u \\ p \end{pmatrix} = \begin{pmatrix} -i\omega \\ |\omega| \\ s'-a|\omega| \end{pmatrix} e^{-|\omega|x}$$

and substituting this in the homogeneous boundary condition, we have

$$t_1|\omega| + t_2(-i\,\omega) + t_3(s'-a|\omega|) = 0 \; .$$

If $t_3 \neq 0$, then

$$s' = |\omega|(a - t_1/t_3) + i\omega\, t_2/t_3 \quad .$$

Since we want $\mathcal{Re}\, s \geq 0$, and $\mathcal{Re}\, s = \mathcal{Re}\, s'$, the condition for a hyperbolic eigensolution is

$$\mathcal{Re}(a-t_1/t_3) + \mathcal{Im}(t_2/t_3) \geq 0 \quad .$$

(We will allow for the t_i to be complex for generality.) If $t_3 = 0$, the condition for a hyperbolic eigensolution is

$$t_1 + i\, t_2 = 0 \qquad \text{or} \qquad t_1 - i\, t_2 = 0 \quad .$$

Now we look for elliptic eigensolutions, (see Definition 4.4). Since $a < 0$, we have $\tilde{v}_0 = 0$ and from equation (5.4), $\tilde{u} = 0$ and $p = p_0 e^{-|\omega| x}$. Rewriting the boundary condition in terms of $\tilde{v}$, $\tilde{u}$, and p we have

$$t_1\tilde{u} + t_2\tilde{v} + (t_3 + (t_1|\omega| - t_2\, i\, \omega)/s')p = \mathcal{O}(\tfrac{-}{s})u + ((\tfrac{\omega}{s})^2)p$$

If $t_3 \neq 0$, then the boundary condition is

$$t_3 p = 0$$

and there is no elliptic eigensolution. If $t_3 = 0$, the boundary condition is

$$(t_1\, \omega - t_2\, i|\omega|)p = 0 \quad .$$

and this gives the same result as the hyperbolic eigensolution for $t_3 = 0$.

Collecting our results, we have that if $a < 0$ there are no eigensolutions of either type when $t_3 \neq 0$, if

$$Re(a - t_1/t_3) + |Im\ t_2 / t_3| < 0 \tag{5.5}$$

and when $t_3 = 0$, if

$$t_1 + i\,t_2 \neq 0 \quad \text{and} \quad t_1 - i\,t_2 \neq 0. \tag{5.6}$$

Case 2. $a > 0$. This corresponds to an inflow boundary. We need two boundary conditions, and without loss of generality we can take them to be

$$\begin{aligned} t_1 u + t_2 v + t_3 p &= g_1(t,y) \\ r_1 u + r_2 v &= g_2(t,y). \end{aligned}$$

We first look for hyperbolic eigensolutions. For $s' \neq |\omega| a$, the general form of a hyperbolic eigensolution is

$$\alpha \begin{pmatrix} -i\omega \\ |\omega| \\ s' - a|\omega| \end{pmatrix} e^{-|\omega| x} + \beta \begin{pmatrix} s' \\ i\omega a \\ 0 \end{pmatrix} e^{-s'x/a} .$$

Substituting this into the homogeneous boundary condition, we see that the condition for an eigensolution to exist is that

$$0 = \det \begin{vmatrix} |\omega| t_1 - i\omega t_2 + (s'-a\,\omega) t_3 & i\omega a t_1 + s' t_2 \\ |\omega| r_1 - i\omega r_2 & i\omega a r_1 + s' r_2 \end{vmatrix}$$

$$= (s'-a|\omega|)(|\omega|(t_1 r_2 - t_2 r_1) + t_3(i\omega a\, r_1 + s' r_2))$$

If $t_3 r_2 \neq 0$, an eigensolution exists if

$$-\mathcal{Re}((t_1 r_2 - t_2 r_1)/t_3 r_2) + |\mathcal{Im}(a\, r_1/r_2)| \geq 0 ,$$

if $t_3 r_2 = 0$, an eigensolution exists if

$$t_1 r_2 - t_2 r_1 \pm i a t_3 r_1 = 0 .$$

For the case when $s' = a|\omega|$, the general form of a hyperbolic eigensolution is

$$(\alpha+\beta x) \begin{pmatrix} -i\omega \\ |\omega| \\ 0 \end{pmatrix} e^{-|\omega| x} + \beta \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix} e^{-|\omega| x} .$$

Substituting this in the homogeneous boundary condition, we arrive at no additional restrictions.

We now look for elliptic eigensolutions. Transforming the boundary conditions, we have

$$t_1 \tilde{u} + t_2 \tilde{v} + (t_3 + (t_1|\omega| - t_2 i\omega)/s')p = O(\tfrac{\omega}{s})\tilde{u} + O(\tfrac{\omega}{s})^2 p$$

$$r_1 \tilde{u} + r_2 \tilde{v} + (r_1|\omega| - r_2 i\omega)/s')p = O(\tfrac{\omega}{s})\tilde{u} + O(\tfrac{\omega}{s})^2 p .$$

We have by Definition 4.4, that $\tilde{v} \neq 0$, $\tilde{u} = 0$, and $p = p_0 e^{-|\omega| x}$.
If $t_3 \neq 0$ the boundary condition is

$$t_3 p = 0$$

$$r_1\tilde{u} + r_2\tilde{v} = 0.$$

So for an eigensolution to exist we must have $r_2 = 0$. If $t_3 = 0$, the boundary condition is

$$t_2\tilde{v} + (t_1|\omega| - t_2\, i\,\omega)p = 0$$

$$r_2\tilde{v} + (r_1|\omega| - r_2\, i\,\omega)p = 0$$

and for an eigensolution to exist we need to have

$$t_1 r_2 - t_2 r_1 = 0 \quad .$$

Collecting the results for Case 2, we have that there are no eigensolutions of either type if, when $t_3 \neq 0$, then

(5.7)
$$r_2 \neq 0, \text{ and}$$
$$Re((t_1 r_2 - t_2 r_1)/t_3 r_2) - |Im(ar_1/r_2)| > 0$$

and for $t_3 = 0$, then

(5.8)
$$t_1 r_2 - t_2 r_1 \neq 0 \quad .$$

Summarizing all this, we have the following theorem:

Theorem 5.1 For the equations 5.1, the boundary conditions give a well-posed initial boundary value problem if and only if they are equivalent to the following boundary conditions:

For $a < 0$,

$$(5.9) \qquad p + (a+c_1)u + c_2 v = g(t,y) \quad \text{with} \quad Re\, c_1 - |Im\, c_2| > 0 \ ,$$

or,

$$(5.10) \qquad c_1 u + c_2\, v = g(t,y) \quad \text{with} \quad c_1^2 + c_2^2 \neq 0 \ ,$$

for $a > 0$,

$$(5.11) \qquad \begin{aligned} c\,u + d\,p &= g_1(t,y) \\ r\,u + v &= g_2(t,y) \ , \end{aligned} \quad \text{with} \quad Re\, c - ad|Im\, r| > 0 \text{ and } d \geq 0 \ .$$

Proof. By Theorem 4.2, the initial boundary value problem is well-posed if and only if there are no eigensolutions of either hyperbolic or elliptic type. If $a < 0$, the conditions that no eigensolutions exist are given by inequalities (5.5) and (5.6). If $t_3 \neq 0$ then without loss of generality, let $t_3 = 1$, then writing $c_2 = t_2$, $c_1 = t_1 - a$ we have the boundary condition (5.9). If $t_3 = 0$, then inequality (5.6) gives the boundary condition, (5.10)

If $a > 0$, the conditions for no eigensolutions are inequalities (5.7) and (5.8). If $t_3 = 0$, then we can choose $t_2 = 0$, $r_2 = 1$ and $t_1 > 0$, this is the boundary conditions (5.11) with $d = 0$. If $t_3 \neq 0$, then also $r_2 \neq 0$, so without loss of generality we can choose $t_3 > 0$, $r_2 = 1$, and $t_2 = 0$. Then (5.7) is the same as (5.11) with $r = r_1$, $c = t_1$, and $t_3 = d$.

We point out that a similar analysis shows that giving the normal component of the velocity and the vorticity $w = v_x - u_y$ at an inflow boundary is a well-posed boundary condition. This boundary condition is that given by Judovič [2].

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